Section 3.3

The Product and Quotient Rules for Differentiation

(1) The Product Rule

(2) The Quotient Rule

Differentiating a Product of Functions

Suppose that $f(x)$ and $g(x)$ are two differentiable functions.
Their product fg is defined by $fg(x) = f(x)g(x)$. **What is $(fg)'(x)$?**

$$\frac{d}{dx}(fg(x)) = \boxed{(fg)'(x)} = \lim_{h\to 0} \frac{\overbrace{f(x+h)g(x+h)}^{fg(x+h)} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h)\overbrace{-f(x+h)g(x) + f(x+h)g(x)}^{=0} - f(x)g(x)}{h}$$

$$= \overbrace{\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}_{=f(x),\ \text{by continuity of } f}\left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right)}^{\text{By Product Law of Limits}} + \overbrace{g(x)\left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right)}^{\text{By Constant Multiple Law of Limits}}$$

$$= \boxed{f(x)g'(x) + g(x)f'(x).}$$

(Note that, f and g are continuous because they are differentiable.)

The Product Rule

To summarize, if $f(x)$ and $g(x)$ are differentiable at $x = a$, then we have just proved that...

(I) The product $fg(x)$ is differentiable at $x = a$.

(II) $(fg)'(a) = f(a)g'(a) + g(a)f'(a)$, or equivalently

$$\frac{d}{dx}(fg(x))\Big|_{x=a} = \left(f(x)\frac{dg}{dx} + g(x)\frac{df}{dx} \right)\Big|_{x=a}$$

(These last two equations say the same thing in Lagrange and Leibniz notation respectively.)

The Product Rule

If $f(x)$ and $g(x)$ are differentiable at $x = a$, then $(fg)(x)$ is differentiable at $x = a$ and

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x).$$

Example I:

(I) $\dfrac{d}{dx}((4x-3)(3x+5))$

(II) $\dfrac{d}{dx}\left(\sqrt{x}(x^2+1)\right)$

(III) $\dfrac{d}{dx}(xe^x)$

Differentiating a Quotient of Functions

$$\frac{d}{dx}\left(\frac{f}{g}(x)\right) = \left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{g(x)f(x+h) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

$$= \frac{g(x)\left(\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right) - f(x)\left(\displaystyle\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)}{g(x)\left(\displaystyle\lim_{h\to 0} g(x+h)\right)}$$

If $f(x)$ and $g(x)$ are differentiable at $x = a$ and $g(a) \neq 0$, then the quotient $\dfrac{f}{g}(x)$ is differentiable at $x = a$.

The Quotient Rule

If $f(x)$ and $g(x)$ are differentiable at $x = a$ and $g(a) \neq 0$, then $\left(\frac{f}{g}\right)(x)$ is differentiable at $x = a$ and

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

Example II:

(I) $\displaystyle \frac{d}{dx}\left(\frac{x^2+1}{\sqrt{x}}\right)$

(II) $\displaystyle \frac{d}{dx}\left(\frac{e^x}{1+e^x}\right)$

Example III, Product Rule

Suppose that

$$f(1) = -2 \qquad g(1) = 4$$
$$f'(1) = -3 \qquad g'(1) = 1$$

(I) **Find $h'(1)$ if $h(x) = 5f(x) - 4g(x)$.**

(II) **Find $h'(1)$ if $h(x) = f(x)g(x)$.**

Example IV, Quotient Rule

Suppose that
$$f(1) = -2 \qquad g(1) = 4$$
$$f'(1) = -3 \qquad g'(1) = 1$$

(I) **Find $h'(1)$ if $h(x) = f(x)/g(x)$.**

(II) **Find $h'(1)$ if $h(x) = \frac{1+g(x)}{1-f(x)}$.**